**Phys 410**
**Fall 2014**
**Lecture #28 Summary**
**9 December, 2014**

We finished discussion of nonlinear mechanics in the context of the harmonically-driven damped pendulum. The bifurcation diagram summarizes the types of motion that are observed under different normalized driving strength $\gamma$. The long-time behavior of the pendulum angle $\phi(t)$ is sampled stroboscopically and plotted on the diagram. The stroboscope period corresponds to the drive period. The regions of period 1, period 2, period 4, and chaotic motion are clearly visible. One surprising result is that periodic motion can be seen in narrow windows of $\gamma$ in the middle of chaotic solutions. Another surprise is that periodic motion can re-appear at larger driving strength, along with other bouts of period doubling and chaos.

We examined the solutions represented as trajectories in a two-dimensional state space described by $(\phi, \dot{\phi})$ as a parametric function of time. These trajectories go on to limit cycle curves for the periodic solutions, and form space-filling curves for chaotic solutions. The running trajectories in which the pendulum winds continuously are hard to represent on bifurcation diagrams and state-space plots. In this case it becomes useful to make Poincare sections that consist of stroboscopically sampled points from state space. These create fractal structures for chaotic solutions.

Finally we discussed some physical realizations of period doubling in diode circuits, neural activity and heat flow through a fluid. There is a direct analogy between the driven damped pendulum and the driven Josephson junction. Certain crystals of high temperature superconductors have an intrinsic Josephson effect between superconducting layers, and such systems act as collections of coupled driven damped pendula, and display an amazing variety of physical phenomena.

We continued our discussion of Special Relativity. Einstein made two postulates:

1) If S is an inertial reference frame and if a second frame S' moves with constant velocity relative to S, then S' is also an inertial reference frame.
2) The speed of light (in vacuum) has the same value *c* in every direction in all inertial reference frames.

We started in to relativistic dynamics with an effort to define relativistic momentum. We expect the laws of physics to have the same form in all reference frames, hence they should be Lorentz invariant. The easiest way to do this is to formulate the laws in terms of 4-vector quantities. The laws of physics should also reduce to familiar Newtonian forms in the small-velocity limit.

Mass is defined to be an invariant quantity (all inertial reference frames agree on its value) and it is equal to the rest mass. Ordinary 3-momentum is not Lorentz invariant. 3-momentum that is conserved in a collision witnessed in one reference frame will not be conserved in another one moving by at relativistic speed (see HW 12, problem 15.54). We need to develop a 4-vector version of momentum. Start with $x^{(4)} = (\vec{x}, ct)$, and consider taking a derivative with respect to time. The problem is that different inertial observers cannot agree on the evolution of time, hence we need a version of time that all observers can agree upon. This would be the proper time interval $dt_0$ which is the differential change in time when the particle of interest is at rest in your reference frame, corresponding to a differential 4-vector of $dx_0^{(4)} = (0, c\,dt_0)$. Comparing the invariant length of this 4-vector to that of a general differential displacement $dx^{(4)} = (\vec{v}, c)dt$ yields $dt_0 = dt/\gamma(v)$, where $\gamma(v) = 1/\sqrt{1-(v/c)^2}$ is the γ-factor associated with particle's velocity as measured in a given reference frame.

With this we can define a *bone-fide* velocity 4-vector that transforms like the space-time 4-vector: $u^{(4)} = \frac{dx^{(4)}}{dt_0} = \gamma(v)(\vec{v}, c)$. We define the associated Lorentz-invariant momentum as $p^{(4)} = m\gamma(v)(\vec{v}, c)$. Note that the 3-vector part of this reduces to the ordinary Newtonian momentum in the $\frac{v}{c} \ll 1$ limit, as required. Note that momentum now carries a fourth component – is this excess baggage or something useful? Recall Noether's theorem (and the idea of ignorable coordinates in Lagrangians), which says that the homogeneity of space implies linear momentum conservation. Likewise the homogeneity of time implies conservation of energy. In this case the time-like component of the momentum 4-vector is defined as relativistic energy $E$ divided by the speed of light, $p^{(4)} = (m\gamma(v)\vec{v}, E/c)$. In other words $E = \gamma(v)mc^2$.

To examine the plausibility of this assignment of the relativistic energy, look at its value in the small-velocity limit $\frac{v}{c} \ll 1$. In this limit $E = \frac{mc^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \cong mc^2 + \frac{1}{2}mv^2 + \cdots$, through a binomial expansion of the denominator. The first term is called the rest energy and is a constant as far as classical Lagrangian mechanics is concerned, hence it plays no role in Newtonian dynamics. The second term is the Newtonian kinetic energy that we have employed since the get-go. Thus this definition of energy reduces to our familiar one in the low-speed Newtonian limit. The relativistic kinetic energy can be written as $T = E - mc^2 = (\gamma(v) - 1)mc^2$.